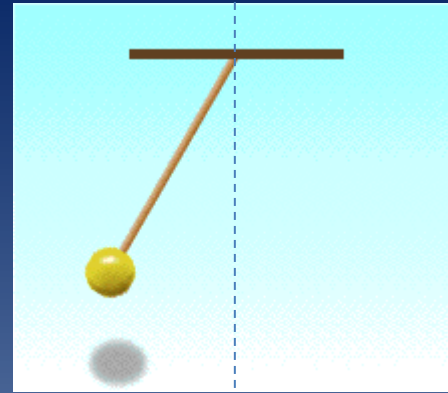
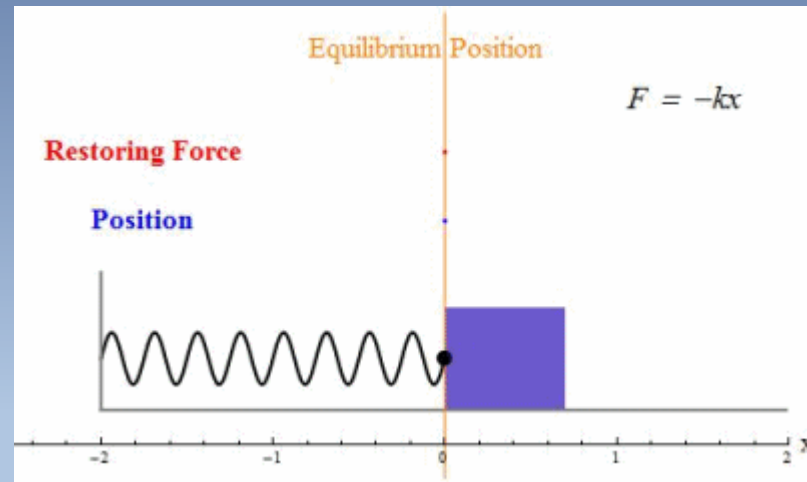


# Periodic motion

## Oscillations



Equilibrium  
position



➔ Any kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine.  
=> This kind of motion, is called **periodic motion** or **oscillation**.

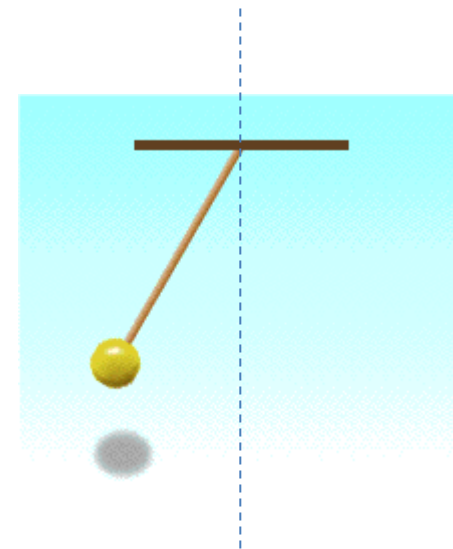
➔ Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

➔ A body that undergoes periodic motion always has a **stable equilibrium position**.

When it is moved away from this position and released, a **force or torque** comes into play to pull it back toward equilibrium (**restoring force/torque**).

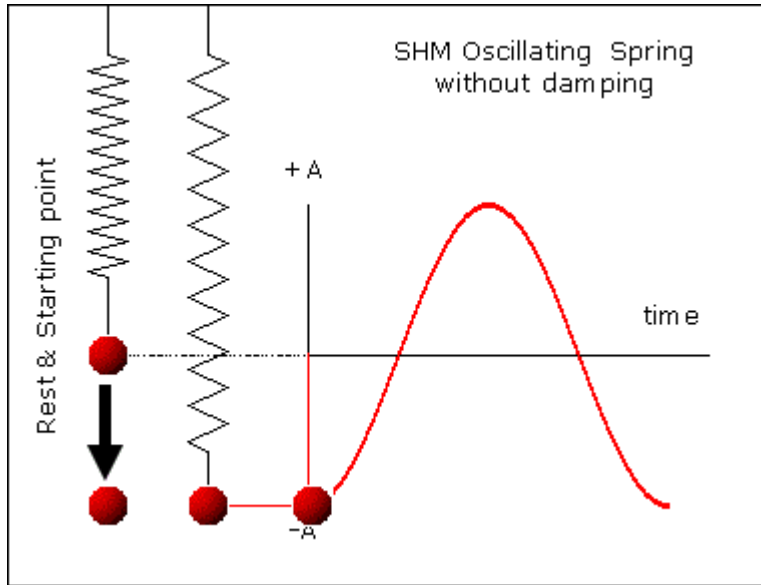
But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium.

Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.



Equilibrium position

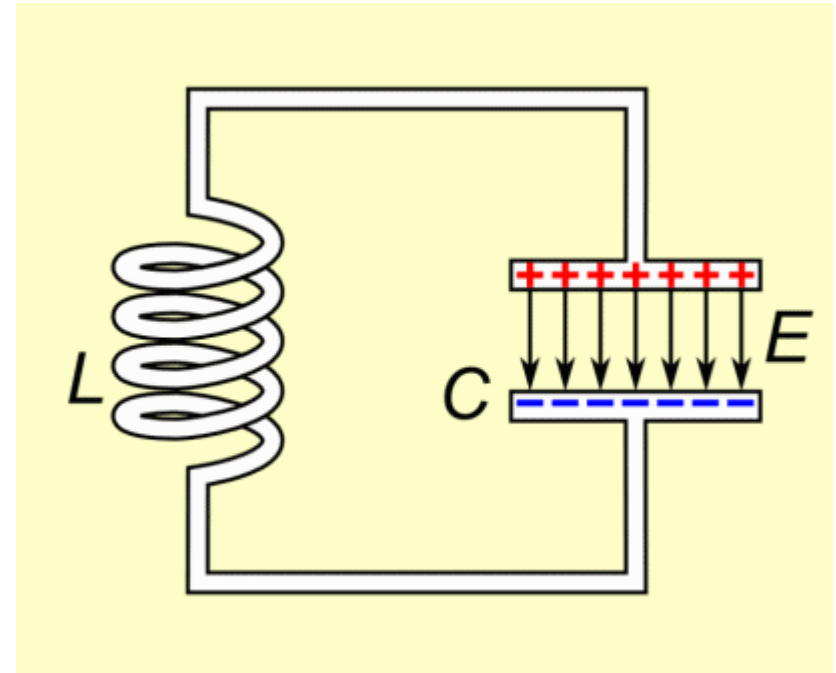
## Mechanic SHM oscillator



$$k \leftrightarrow 1/C$$

$$m \leftrightarrow L$$

## Electrical oscillator LC



Continuous conversion

$$K \leftrightarrow U$$

$$mv^2 / 2 \leftrightarrow kx^2 / 2$$

$$E_m \leftrightarrow E_{el}$$

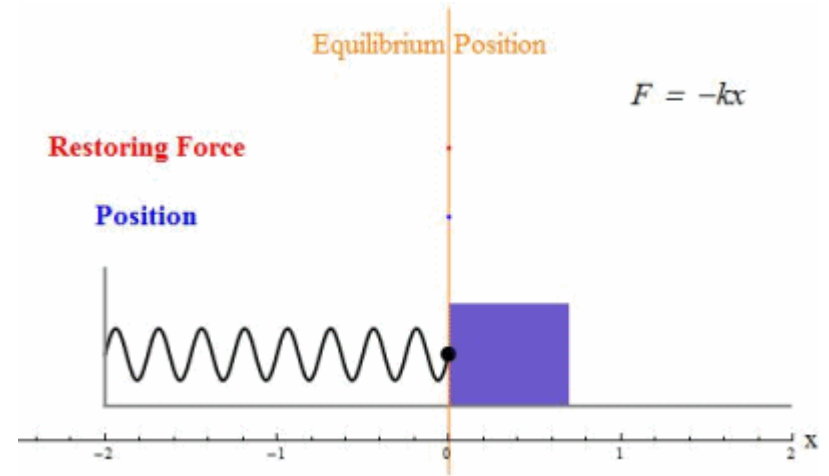
$$LI^2 / 2 \leftrightarrow Q^2 / (2C)$$

## DESCRIBING OSCILLATIONS

When the body is displaced from its equilibrium position at  $x=0$  the spring exerts a **restoring force** back toward the equilibrium position.

⇒ **Oscillation**

= periodic motion around the equilibrium



## Amplitude, Period, Frequency, and Angular Frequency

**The amplitude  $A$** , of the motion = *is the maximum magnitude of displacement from equilibrium*

⇔  **$A$  = maximum value of  $|x|$** . It is always positive.  $[A] = \text{m}$

A complete vibration, or **cycle**, is one complete round trip—say, from  $A$  to  $-A$  and back to  $A$ , or from  $O$  to  $A$ , back through  $O$  to  $-A$  and back to  $O$ . The motion from one side to the other (say,  $-A$  to  $A$ ) is a **half-cycle**, not a whole cycle.

**The period  $T$**  = *the time for one cycle*. It is always positive.

$[T] = \text{s}$ , sometimes expressed as “seconds per cycle.”

**The frequency  $f$**  = *the number of cycles in a unit of time*. It is always positive.

$[f] = \text{the hertz: } 1 \text{ hertz} = 1 \text{ Hz} = 1 \text{ cycle} \cdot \text{s}^{-1} = 1 \text{ s}^{-1}$

$$f = 1/T$$

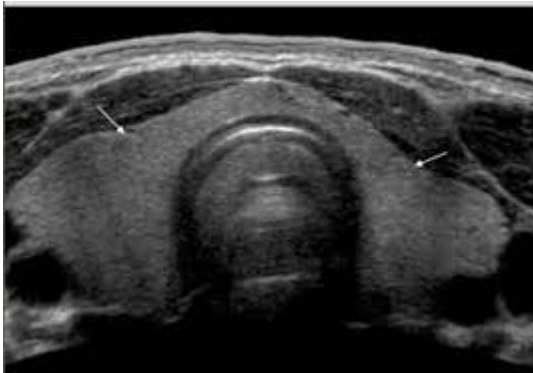
The **angular frequency**  $\omega$  is  $2\pi$  times the frequency:  $\omega = 2\pi f = \frac{2\pi}{T}$   $[\omega]=\text{rad/s}$

**Example:**

An ultrasonic transducer used for medical diagnosis oscillates at  $6.7 \text{ MHz} = 6.7 \times 10^6 \text{ Hz}$ . How long does each oscillation take, and what is the angular frequency?

$$T = \frac{1}{f} = \frac{1}{6.7 \times 10^6 \text{ Hz}} = 1.5 \times 10^{-7} \text{ s} = 0.15 \mu\text{s}$$
$$\omega = 2\pi f = 2\pi(6.7 \times 10^6 \text{ Hz})$$
$$= (2\pi \text{ rad/cycle})(6.7 \times 10^6 \text{ cycle/s})$$
$$= 4.2 \times 10^7 \text{ rad/s}$$

This is a very rapid vibration, with large  $f$  and  $\omega$  and small  $T$ . A slow vibration has small  $f$  and  $\omega$  and large  $T$ .



*thyroid sonography*

## LEARNING GOALS

*By studying this chapter, you will learn:*

- How to describe oscillations in terms of amplitude, period, frequency, and angular frequency.
- How to do calculations with simple harmonic motion, an important type of oscillation.
- How to use energy concepts to analyze simple harmonic motion.
- How to apply the ideas of simple harmonic motion to different physical situations.
- What determines how rapidly an oscillation dies out.
- How a driving force applied to an oscillator at the right frequency can cause a very large response, or resonance.

### Plan:

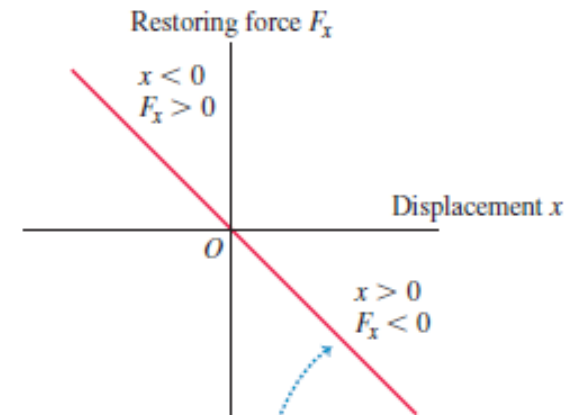
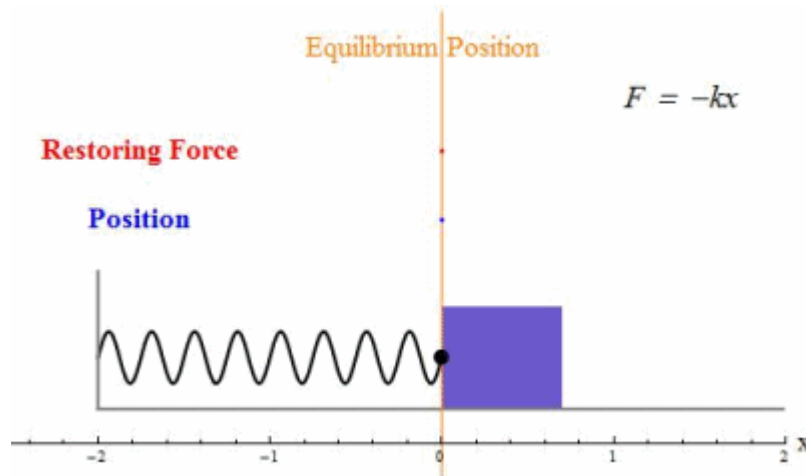
- Simple Harmonic Motion (SHM): *ideal case, no dissipation*
- Damped oscillations: *realistic case, dissipation*
- Forced Oscillations and resonance: *periodic force on damped oscillator to keep  $A=ct$*
- ***Electric analogy:*** the RLC oscillator

## (I) The simple harmonic motion (SHM)

The simplest kind of oscillation occurs when the restoring force  $F_x$  is *directly proportional* to the *displacement from equilibrium*  $x$ .

$$F_x = -kx \quad (\text{restoring force exerted by an ideal spring})$$

Hooke's law



The restoring force exerted by an idealized spring is directly proportional to the displacement (Hooke's law,  $F_x = -kx$ ): the graph of  $F_x$  versus  $x$  is a straight line.

The acceleration in SHM:  $a_x = d^2x/dt^2 = F_x/m$

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad (\text{simple harmonic motion})$$

The *minus sign* means the *acceleration* and *displacement* always have opposite signs. The acceleration is *not* constant in time.

$$\rightarrow \frac{d^2 x}{dt^2} + \frac{k}{m} x = 0$$

2<sup>nd</sup> order differential equation whose solution will be  $x=x(t)$

notation  $\frac{k}{m} = \omega^2$   $\rightarrow$

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0$$

Frequency and period:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (\text{simple harmonic motion})$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \quad (\text{simple harmonic motion})$$

The **period**  $T$  and **frequency**  $f$  of simple harmonic motion are completely determined by the mass  $m$  and the force constant  $k$ .

*In simple harmonic motion the period and frequency do not depend on the amplitude  $A$ .*



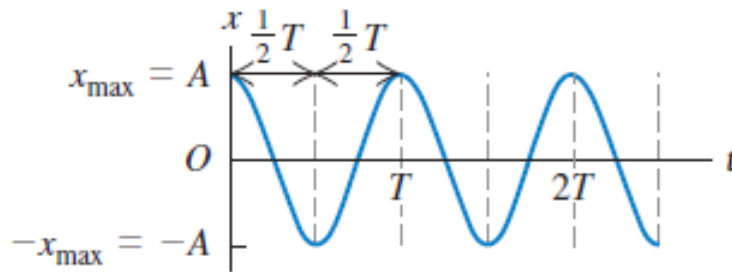
# Displacement, Velocity, and Acceleration in SHM

The solution of the differential equation of the SHM is:

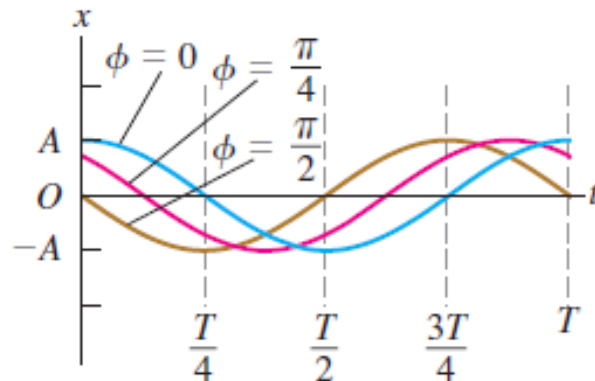
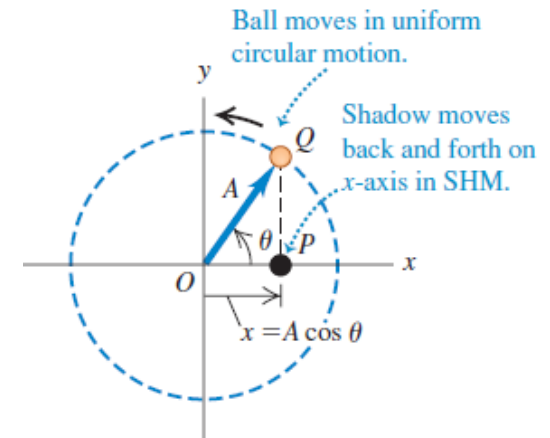
$$x = A \cos(\omega t + \phi) \quad (\text{displacement in SHM})$$

$$\omega = \sqrt{k/m}.$$

The displacement  $x$  is a periodic function of time, as expected for SHM.



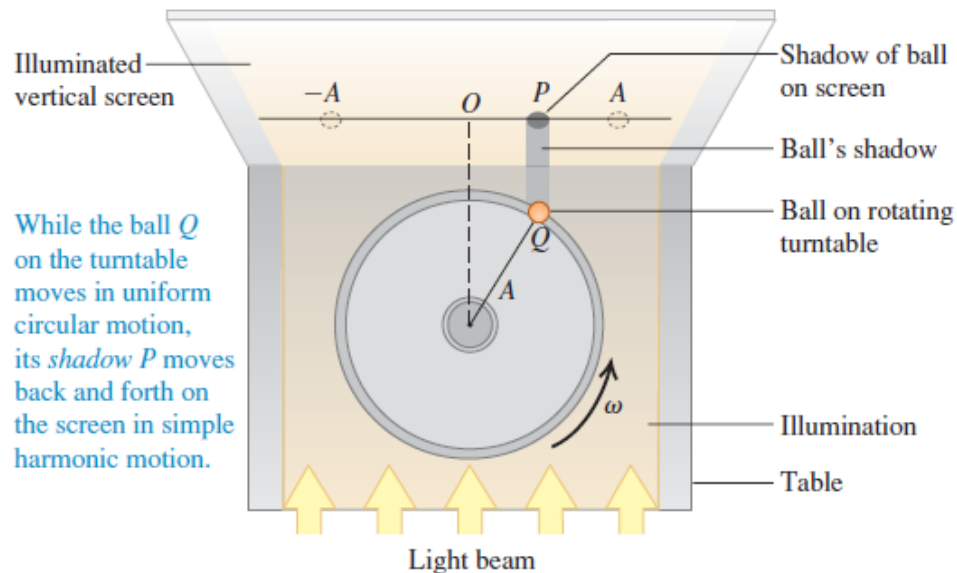
The constant  $\phi$  is called the **phase angle**. It tells us at what point in the cycle the motion was at  $t=0$  (equivalent to where around the circle the point  $Q$  was at  $t=0$  –see next slide).



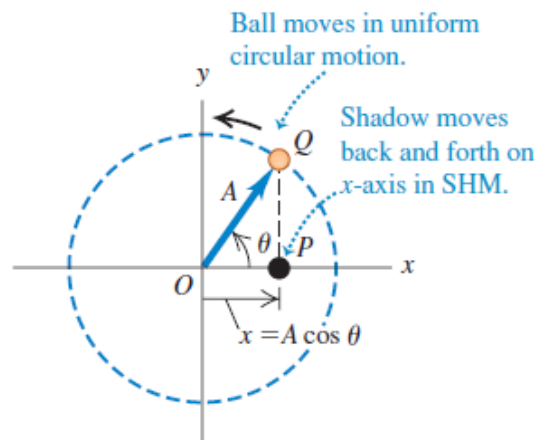
# Relating uniform circular motion and simple harmonic motion.

## Phasor representation.

(a) Apparatus for creating the reference circle



(b) An abstract representation of the motion in (a)



(a) Relating uniform circular motion and simple harmonic motion.

(b) The ball's shadow moves exactly like a body oscillating on an ideal spring.

At time  $t$  the vector  $OQ$  from the origin to the reference point  $Q$  makes an angle  $\theta$  with the positive  $x$ -axis. As the point  $Q$  moves around the reference circle with constant angular speed the vector  $OQ$  rotates with the same angular speed  $\omega$ . Such a **rotating vector** is called a **phasor**.

Simple harmonic motion is the projection of uniform circular motion onto a diameter.  $x = A \cos \theta$

## Velocity, and Acceleration in SHM

functions of time for a harmonic oscillator

=> by taking derivatives of  $x(t)$  with respect to time:

$$x = A \cos(\omega t + \phi)$$

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

Oscillates between  $v_{\max} = +\omega A$  and  $v_{\min} = -\omega A$

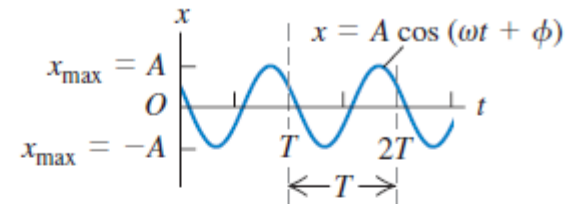
$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

Oscillates between  $a_{\max} = +\omega^2 A$  and  $a_{\min} = -\omega^2 A$

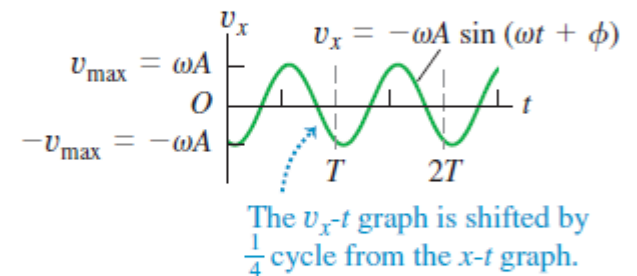
**Obs:**

$$\begin{cases} x(t) = 0 & v = v_{\max} & a_x = 0 \\ x(t) = \pm A & v = 0 & a_x = a_{\max} \end{cases}$$

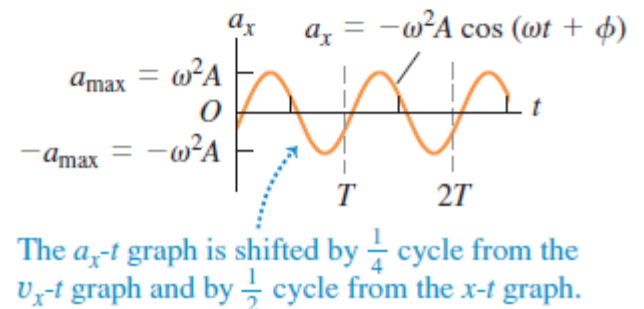
(a) Displacement  $x$  as a function of time  $t$



(b) Velocity  $v_x$  as a function of time  $t$



(c) Acceleration  $a_x$  as a function of time  $t$



## Energy in Simple Harmonic Motion

We can learn even more about simple harmonic motion by using energy considerations.

The kinetic energy in the SHM:  $K = \frac{1}{2}mv_x^2$

$$x = A \cos(\omega t + \phi)$$

$$\frac{k}{m} = \omega^2$$

The potential energy in the SHM:  $U = \frac{1}{2}kx^2$

$$v_x = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

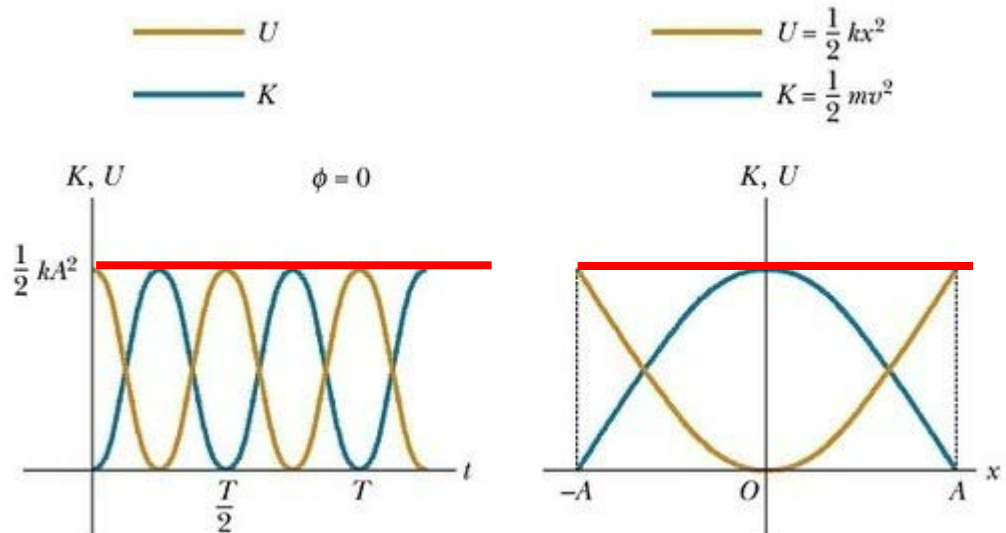
➔ The total energy  $E=K+U$

$$\begin{aligned} E &= \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}m[-\omega A \sin(\omega t + \phi)]^2 + \frac{1}{2}k[A \cos(\omega t + \phi)]^2 \\ &= \frac{1}{2}kA^2 \sin^2(\omega t + \phi) + \frac{1}{2}kA^2 \cos^2(\omega t + \phi) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

$E$  will be constant in time (conserved)

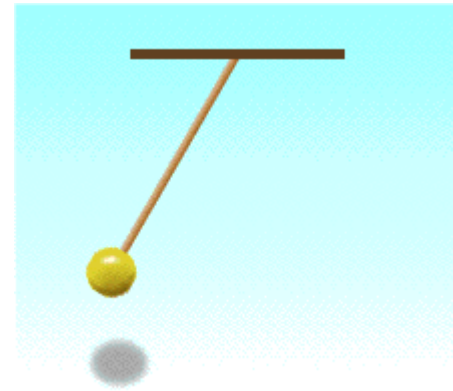
$A$  constant in time

Elastic force = conservative force  
 $\Rightarrow$  Energy conservation, continuous conversion from  $K$  to  $U$



## Application 1. The Simple Pendulum

A **simple pendulum** is an idealized model consisting of a point mass suspended by a massless, unstretchable string. When the point mass is pulled to one side of its straight-down equilibrium position and released, it oscillates about the equilibrium position.



The restoring force is the tangential component of the net force:

$$F_{\theta} = -mg \sin \theta \quad \text{For small } \theta, \sin \theta \approx \theta$$

$$F_{\theta} = -mg\theta = -mg \frac{x}{L} \quad \text{or}$$

$$F_{\theta} = -\frac{mg}{L}x$$

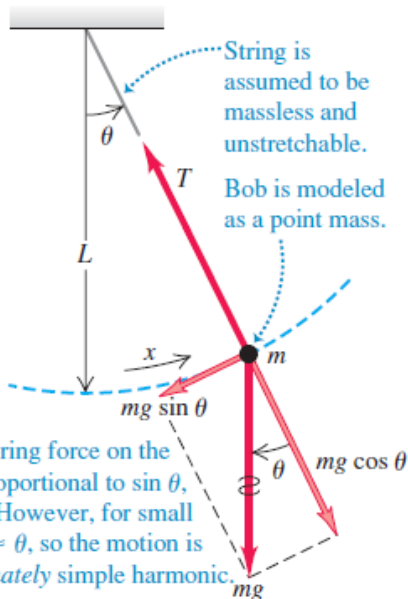
Is of elastic force type  $F=-kx$

$$\Rightarrow k = mg/L.$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \quad \text{(simple pendulum, small amplitude)}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi \sqrt{\frac{L}{g}}$$



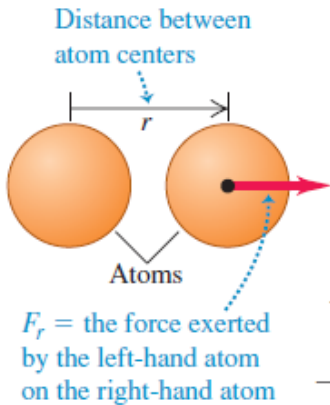
$T$  depends on  $L$  and  $g$  but not on  $m$

Pendulum clock beats second on Earth but not on Moon

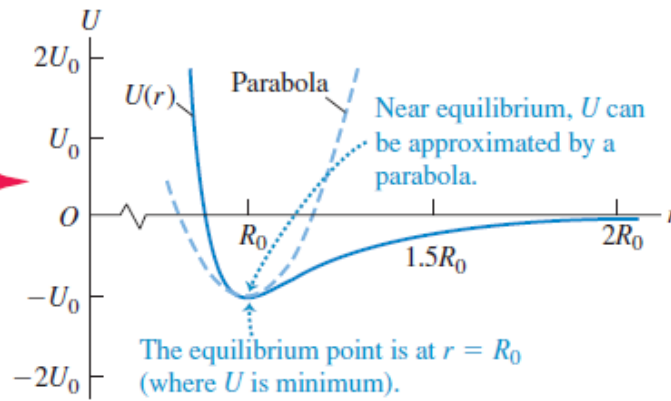
## Application 2. Vibrations of Molecules/atoms in solids

When two atoms are separated from each other by a few atomic diameters, they can exert attractive forces on each other. But if the atoms are so close to each other that their electron shells overlap, the forces between the atoms are repulsive. Between these limits, there can be an equilibrium separation distance at which two atoms form a *molecule*. If these atoms are displaced slightly from equilibrium, they will oscillate.

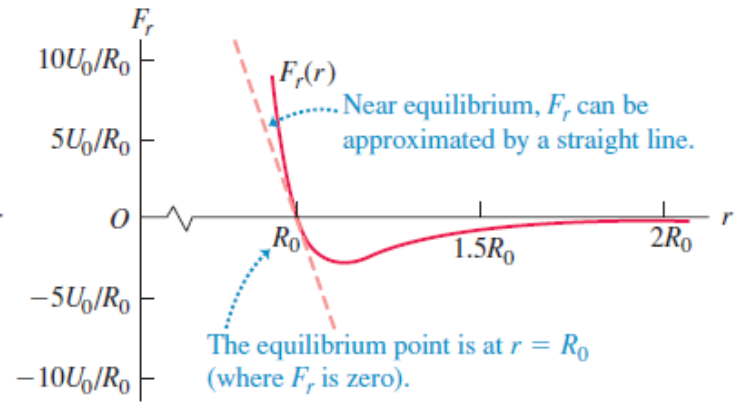
(a) Two-atom system



(b) Potential energy  $U$  of the two-atom system as a function of  $r$



(c) The force  $F_r$  as a function of  $r$



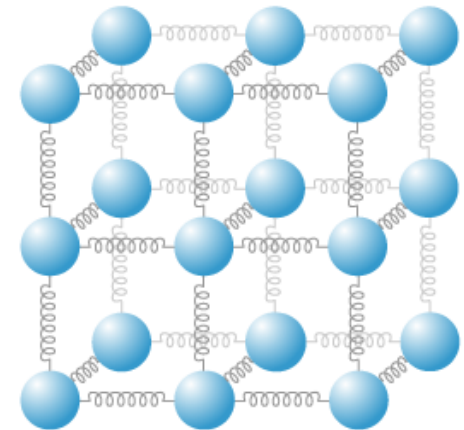
$$U(r) = -k(r-r_0)^2/2 \Rightarrow F(r) = -dU/dr = -k(r-r_0)$$

$\Rightarrow$  restoring force  $\Rightarrow$  oscillation around equilibrium  $r=r_0$

### Extrapolation:

vibration of atoms in solids at finite temperature.

*See next term quantum mechanics*



---

## A simple pendulum

Find the period and frequency of a simple pendulum 1.000 m long at a location where  $g = 9.800 \text{ m/s}^2$ .

**EXECUTE:**

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{1.000 \text{ m}}{9.800 \text{ m/s}^2}} = 2.007 \text{ s}$$

$$f = \frac{1}{T} = \frac{1}{2.007 \text{ s}} = 0.4983 \text{ Hz}$$

**EVALUATE:** The period is almost exactly 2 s. When the metric system was established, the second was *defined* as half the period of a 1-m simple pendulum. This was a poor standard, however, because the value of  $g$  varies from place to place.

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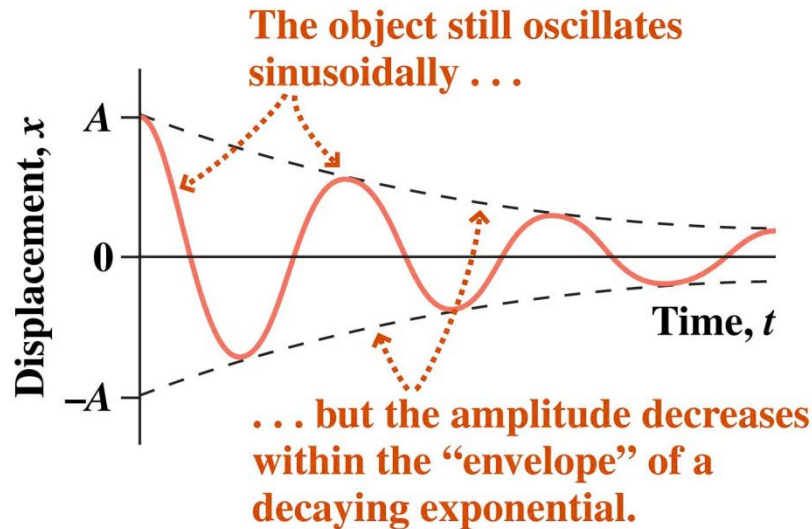


## (II) Damped Oscillations

The idealized oscillating systems (SHM) are frictionless. There are no nonconservative forces, the total mechanical energy is constant, and a system set into motion continues oscillating forever with no decrease in amplitude.

Real-world systems always have some dissipative forces: *viscosity, friction...*

=> **dissipation**, oscillations die out with time unless we replace the dissipated mechanical energy. *A mechanical pendulum clock continues to run because potential energy stored in the spring or a hanging weight system replaces the mechanical energy lost due to friction in the pivot and the gears.*





The decrease in amplitude caused by dissipative forces is called **damping**, and the corresponding motion is called **damped oscillation**.

The simplest case to analyze in detail is a simple harmonic oscillator with a **frictional damping force that is directly proportional to the velocity** of the oscillating body. This behavior occurs in friction involving viscous fluid flow, such as in shock absorbers or sliding between oil-lubricated surfaces.

We then have an additional force on the body due to friction:  $F_x = -bv_x$

where  $v_x = \frac{dx}{dt}$

is the velocity and  **$b$  is a constant that describes the strength of the damping force**. The *negative sign* shows that the force is always opposite in direction to the velocity.

The net force on the body will be:

$$\sum F_x = -kx - bv_x$$

And the Newton's second law:

$$-kx - bv_x = ma_x \quad \text{or}$$

$$-kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}$$

2<sup>nd</sup> order differential equation of damped oscillator

If the damping force is relatively small, the motion is described by:

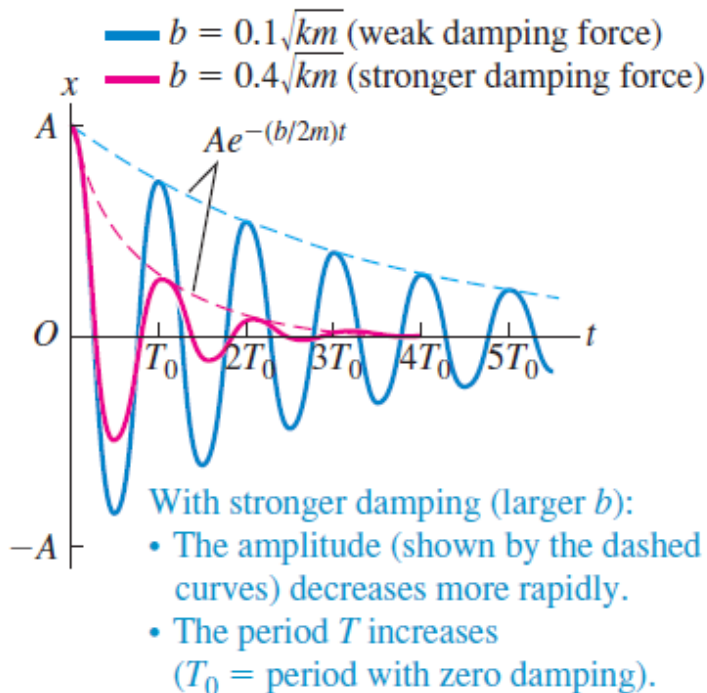
$$x = Ae^{-(b/2m)t} \cos(\omega't + \phi) \quad (\text{oscillator with little damping})$$

$$x = Ae^{-(b/2m)t} \cos(\omega't + \phi) \quad (\text{oscillator with little damping})$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

The angular frequency  $\omega'$  of oscillation

Graph of displacement versus time for an oscillator with little damping and with phase angle  $\Phi = 0$ . The curves are for two values of the damping constant  $b$ .



**Obs:**

$$A(t) = Ae^{-(b/2m)t} = Ae^{-\delta t}$$

is not constant but exponentially decaying in time  
 larger  $b$  (or  $\delta$ ), larger decay

## Damping regimes

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \delta^2} < \omega_0$$

1.

$$\omega' = 0 \quad \text{if} \quad \frac{k}{m} = \frac{b^2}{4m^2} \quad \text{or} \quad b = \sqrt{km}$$

the condition is called **critical damping**. The system no longer oscillates but returns to its equilibrium position without oscillation when it is displaced and released.

2.

If  $b > \sqrt{km}$  the condition is called **overdamping**.

Again there is no oscillation, but the system returns to equilibrium more slowly than with critical damping.

For the overdamped case the solutions of 2<sup>nd</sup> order diff. eq of damped oscillator have the form:

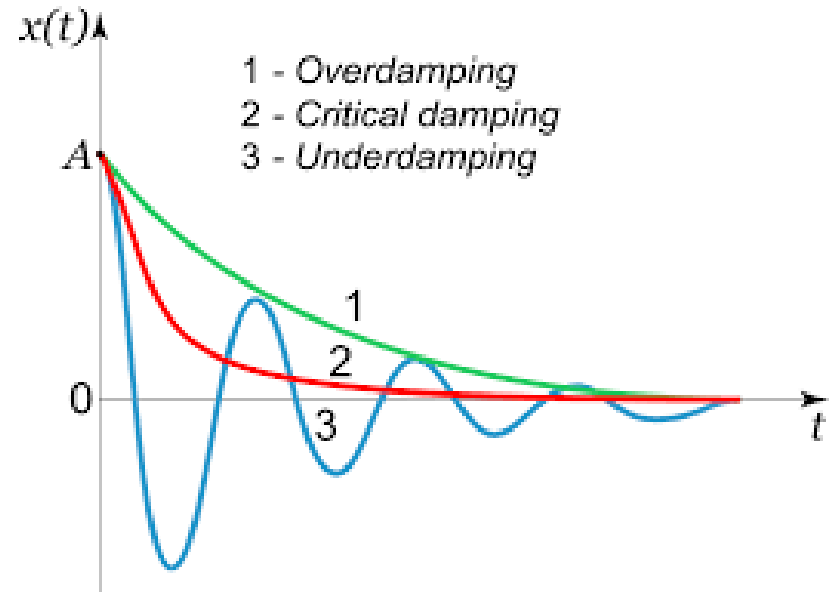
$$x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t}$$

where  $C_1$  and  $C_2$  are constants that depend on the initial conditions and  $a_1$  and  $a_2$  are constants determined by  $m$ ,  $k$ , and  $b$ .

3.

If  $b < \sqrt{km}$  than the critical value, the condition is called **underdamping**.

The system oscillates with **steadily decreasing amplitude**.



## Energy in Damped Oscillations

In damped oscillations the damping force is nonconservative; the mechanical energy of the system is not constant but decreases continuously, approaching zero after a long time. To derive an expression for the rate of change of energy, we first write an expression for the total mechanical energy  $E$  at any instant:

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$

To find the rate of change of this quantity, we take its time derivative:  $\frac{dE}{dt} = mv_x \frac{dv_x}{dt} + kx \frac{dx}{dt}$

But  $dv_x/dt = a_x$  and  $dx/dt = v_x$ , so

$$\frac{dE}{dt} = v_x(ma_x + kx) \quad \Rightarrow \quad ma_x + kx = -b dx/dt = -bv_x,$$

$$\Rightarrow \frac{dE}{dt} = v_x(-bv_x) = -bv_x^2$$

Is always negative  $\Rightarrow$  as the body moves, the energy decreases, though not at a uniform rate.

The term:  $-bv_x^2 = (-bv_x)v_x$  (force times velocity)

Is the rate at which the damping force does (negative) work on the system (that is, the damping *power*). This equals the rate of change of the total mechanical energy of the system.

Similar behavior occurs in electric circuits containing inductance **L**, capacitance **C**, and resistance **R**. There is a natural frequency of oscillation, and the resistance plays the role of the damping constant  $b$ .

## Important physical quantities describing damped oscillations

Characteristic for any type of damped oscillatory phenomena, independently on their nature: mechanical, electrical, electromagnetic. (see later the RLC oscillator).

### Damping logarithmic decrement

$$\theta = \ln \frac{A(t)}{A(t+T)} = \ln e^{\frac{b}{2m}T} = \frac{b}{2m}T = \delta T$$

$\delta$  = damping parameter

**Relaxation time** = the time after which the amplitude decays e times  
e=2,71828 the basis of natural logarithms

$$A(t+\tau) = \frac{A(t)}{e} \Rightarrow \tau = \frac{1}{\delta}$$

**The quality factor Q** = defines the speed of energy loss due to dissipation

⇔ a measure of the relationship between stored energy and rate of energy dissipation

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy dissipated per cycle}} = 2\pi \frac{E(t)}{E(t) - E(t+T)} = 2\pi \frac{E}{\Delta E}$$

If introducing the value for E(t) =>

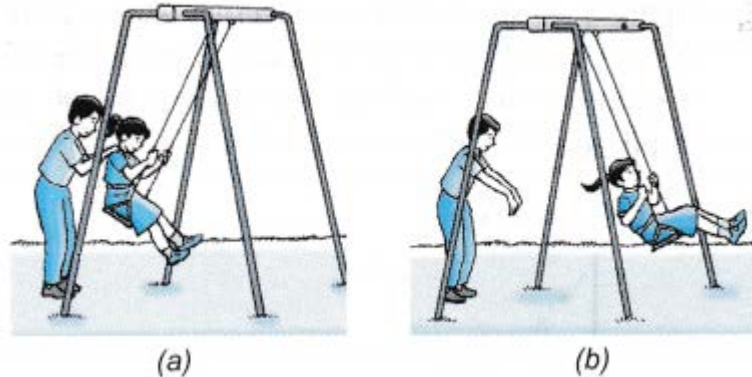
$$Q = \frac{2\pi}{T} \tau = \omega_0 \tau = \frac{\omega_0}{\delta} = \sqrt{\frac{km}{b}}$$

if  $\delta > \omega_0$  (or  $b > \sqrt{km}$ )  $\Rightarrow Q < 1$  unharmonic oscillations (overdamping)

if  $\delta < \omega_0$  (or  $b < \sqrt{km}$ )  $\Rightarrow Q > 1$  harmonic oscillations (underdamping)

### (III) Forced Oscillations and resonance

A damped oscillator left to itself will eventually stop moving altogether. But we can maintain a constant-amplitude oscillation by applying a force that varies with time in a periodic or cyclic way, with a definite period and frequency.



#### Damped Oscillation with a Periodic Driving Force

If we apply a periodically varying driving force with angular frequency  $\omega_d$  to a damped harmonic oscillator, the motion that results is called a **forced oscillation** or a *driven oscillation*.

In a forced oscillation, the angular frequency with which the mass oscillates is equal to the driving angular frequency  $\omega_d$ .

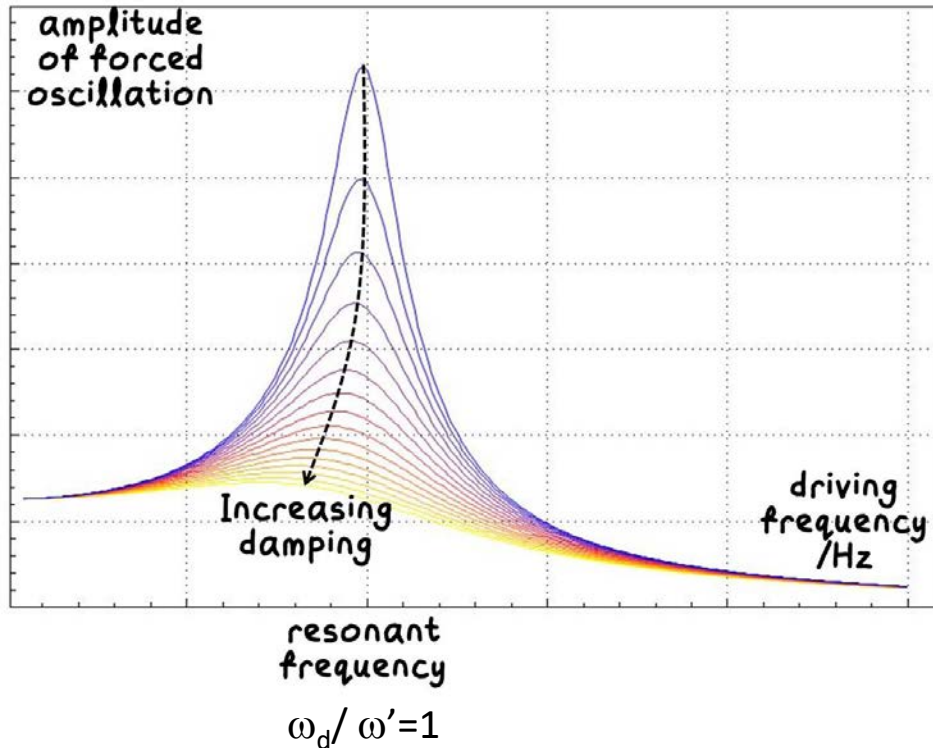
It is different from the motion that occurs when the system is simply displaced from equilibrium and then left alone, in which case the system oscillates with a **natural angular frequency**  $\omega'$  determined by  $m$ ,  $k$ , and  $b$ .

**The amplitude of oscillations is constant but depends on  $\omega_d/\omega'$ .**

If we apply a periodic force (e.g. sinusoidal) to a damped oscillator:

$$F(t) = F_{\max} \cos \omega_d t$$

If we vary the frequency of the driving force, the amplitude of the resulting forced oscillation varies in an interesting way:



the amplitude goes through a sharp peak as the driving angular frequency nears the natural oscillation angular frequency  $\omega' \Rightarrow$  **RESONANCE**

When the damping is increased (larger  $b$ ), the peak becomes broader and smaller in height and shifts toward lower frequencies.

From the second principle of dynamics:

$$\sum F = ma_x = m \frac{d^2 x}{dt^2} \Rightarrow -kx - b v_x - F_{\max} \cos \omega_d t = m \frac{d^2 x}{dt^2} \Rightarrow -kx - b \frac{dx}{dt} - F_{\max} \cos \omega_d t = m \frac{d^2 x}{dt^2}$$

$$\Rightarrow \frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x + \frac{F_{\max}}{m} \cos \omega_d t = 0$$

The differential equation of forced oscillator

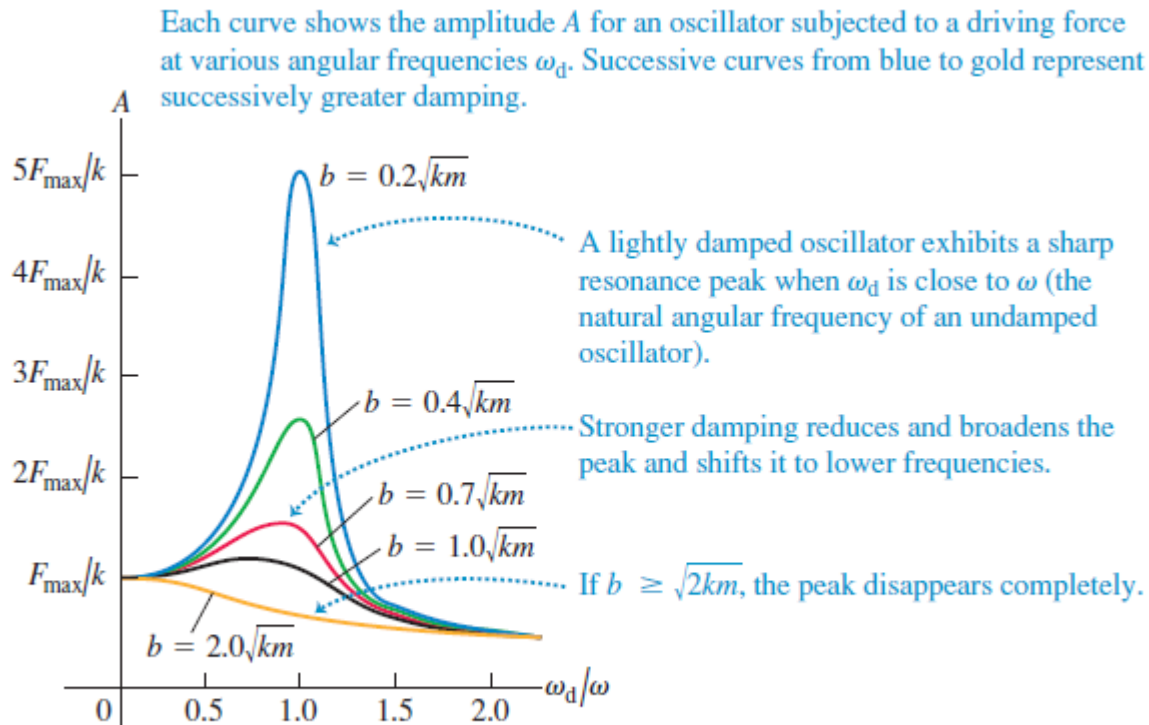
Solving the 2<sup>nd</sup> order differential equation, one gets:  $x(t) = A \cos(\omega_d t + \phi)$

with

$$A = \frac{F_{\max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \quad (\text{amplitude of a driven oscillator})$$

$$\tan \phi = \frac{\frac{b}{m} \omega_d}{\omega_d^2 - \omega_0^2}$$

When  $k - m\omega_d^2 = 0$ , the first term under the radical is zero, so  $A$  has a maximum near  $\omega_d = \sqrt{k/m}$ . The height of the curve at this point is proportional to  $1/b$ ; the less damping, the higher the peak. At the low-frequency extreme, when  $\omega_d = 0$ , we get  $A = F_{\max}/k$ . This corresponds to a *constant* force  $F_{\max}$  and a constant displacement  $A = F_{\max}/k$  from equilibrium, as we might expect.



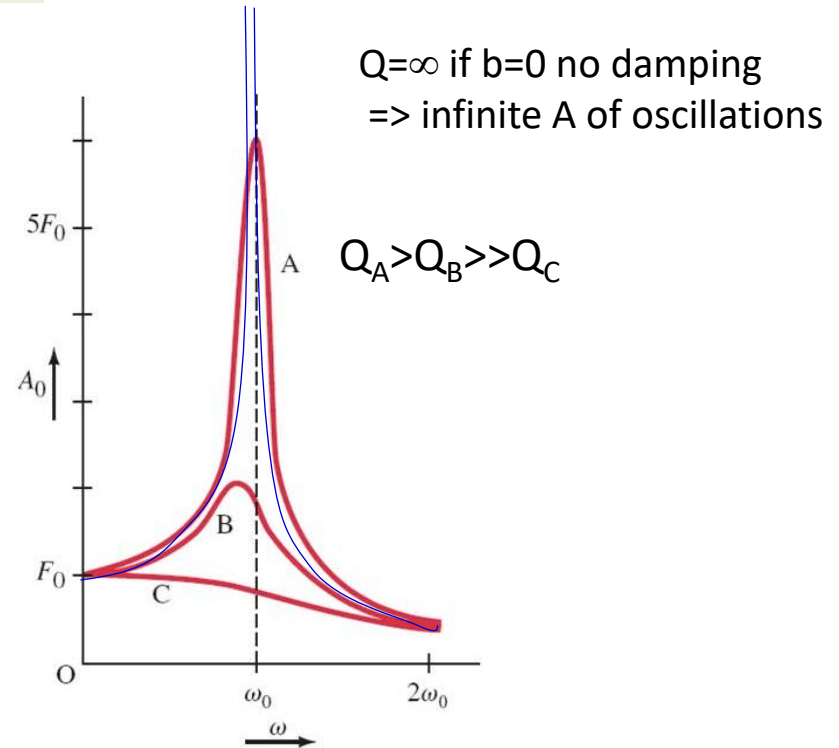


## Relationship with the quality factor Q

The fact that there is an amplitude peak at driving frequencies close to the natural frequency of the system is called **resonance**.

The width of the resonant peak can be characterized by the  $Q$  factor:

$$Q = \frac{m\omega_0}{b}$$



## Resonance and Its Consequences

For low damping forced oscillators close to resonance, the amplitude of oscillations becomes extremely large ( $Q=\infty$  if  $b=0$  no damping => **infinite amplitude A** of oscillations).

### **Destructive effects:**

if  $A_{\text{resonance}} > \text{plasticity limit}$  => mechanical fracture of oscillators.  
or in electric circuits the current may attain dangerously large values

*A company of soldiers once destroyed a bridge by marching across it in step; the frequency of their steps was close to a natural vibration frequency of the bridge, and the resulting oscillation had large enough amplitude to tear the bridge apart. Ever since, marching soldiers have been ordered to break step before crossing a bridge. Some years ago, vibrations of the engines of a particular airplane had just the right frequency to resonate with the natural frequencies of its wings. Large oscillations built up, and occasionally the wings fell off.*

**Positive effects:** at resonance the transfer of energy is maximum.  
see other applications when studying waves

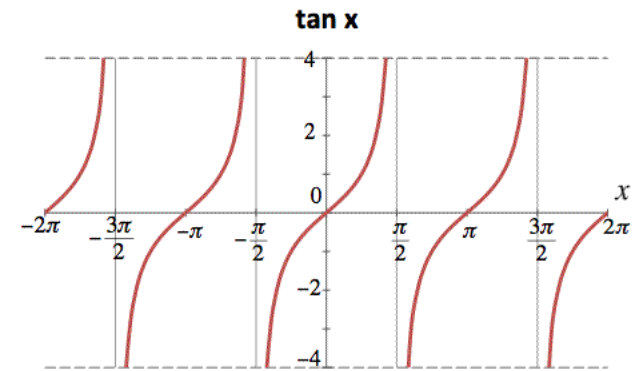
### **Tuning radio/TV stations**

Resonance also occurs in electric circuits (see 2nd term) a tuned circuit in a radio or television receiver responds strongly to waves having frequencies near its resonant frequency, and this fact is used to select a particular station and reject the others.

## Discussion about the phase

(see movie).

$$\tan \phi = \left( \frac{b}{m} \right) \frac{\omega_d}{\omega_d^2 - \omega_0^2}$$



1. **Small  $\omega_d$  and  $\omega_d \ll \omega_0$**   $\Rightarrow \tan \phi \rightarrow 0$  ;  $\phi \rightarrow 0$  the phase angle tends to zero

The low frequency response is in phase with the driving force. The oscillator, through its elastic constant  $k$  controls the response, the body is displaced back and forth by the external force acting against the elastic force.

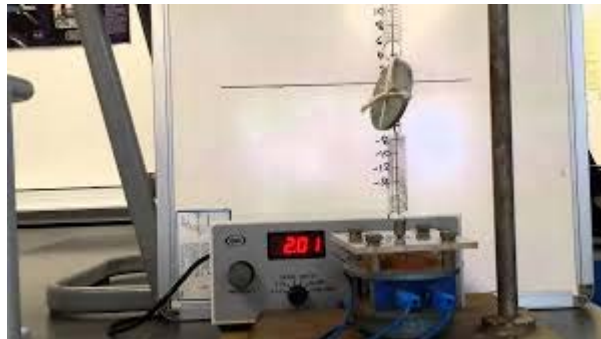
2.  **$\omega_d = \omega_0$**   $\Rightarrow \tan \phi \rightarrow \infty$  ;  $\phi = \pi/2$  ;

the force is phase-shifted with respect to the elongation with  $\pi/2$   
 $\Rightarrow$  in phase with the velocity. In order to have a maximum power transfer the force has to be maximum when the velocity of the oscillator is maximum.

3.  **$\omega_d > \omega_0$**   $\Rightarrow \tan \phi \rightarrow 0$  ;  $\phi = \pi$  ;  $\Rightarrow \mathbf{A} \rightarrow \mathbf{F}_{\max} / m\omega_d^2$

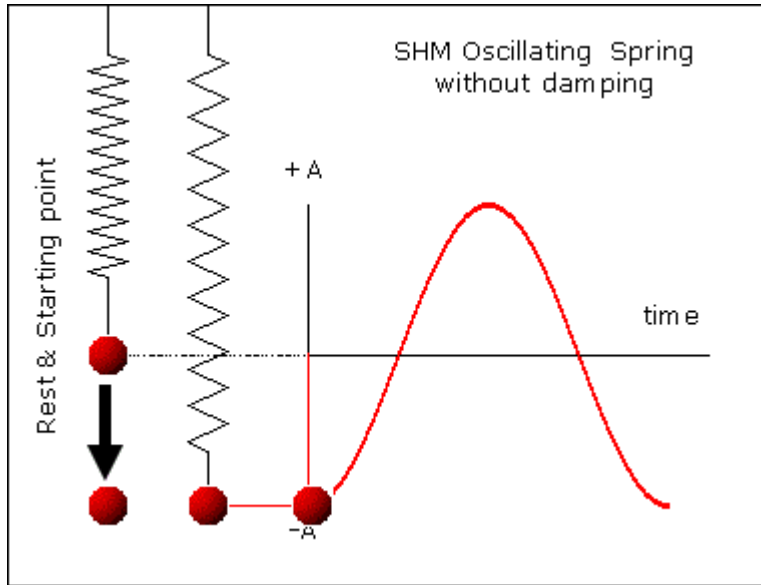
the amplitude decays with  $\omega_d$ , the response of the oscillator is controlled by the inertia of the system ( $m$ ) and  $m$  respond as a free object being displaced back and forth by the force  
 $\Rightarrow$  the elongation is always delayed (in antiphase) with respect to the driving force.

## Some videos about forced oscillations



# MECHANIC AND ELECTROMAGNETIC OSCILLATIONS

## Mechanic SHM oscillator



$$k \leftrightarrow 1/C$$

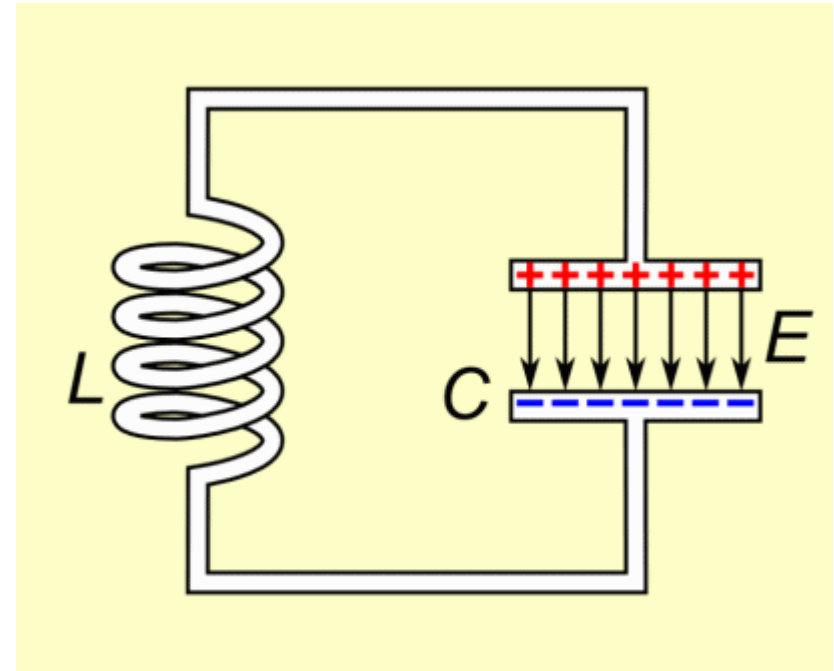
$$m \leftrightarrow L$$

$$K \leftrightarrow U$$

Continuous conversion

$$mv^2 / 2 \leftrightarrow kx^2 / 2$$

## Electrical oscillator LC



$$E_m \leftrightarrow E_{el}$$

$$LI^2 / 2 \leftrightarrow q^2 / (2C)$$

**Goal: study the**

Damped oscillator

Dissipation  $b$ :  $F = -bv$

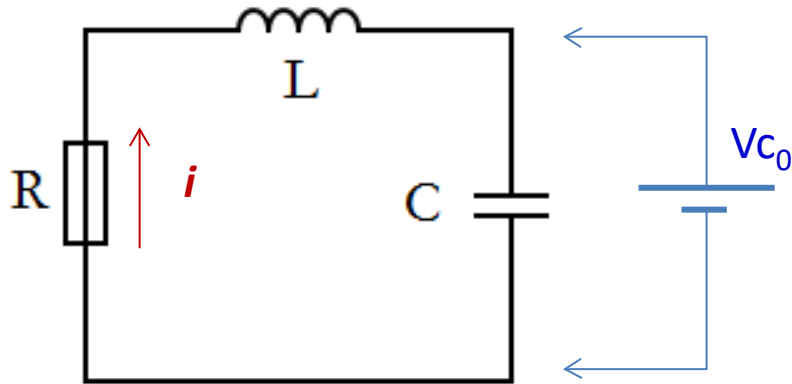
Realistic case

**Damped RLC oscillator**

Dissipation  $R \leftrightarrow b$ : *dissipation power*  $= -RI$

The SHO constitutes an example of periodical motion of major importance because it serves as an exact or approximate model for many problems of classic and quantum physics.

Based on analogy with SHM, we are going to analyze the variation in time of the electric charge in an RLC serial circuit.



We assume that at instant  $t=0$  the capacitor  $C$  has been charged with the total amount of charge  $q_0$  (e.g. using a source  $V_{c_0}$ , then the system is let free.

We do not have any source in the circuit, so the sum of voltages on the closed loop (on the  $R$ ,  $L$ ,  $C$ ) has to be zero.

$$\boxed{V_C + V_R + V_L = 0}$$

$$\text{with } \begin{cases} V_C = \frac{q}{C} \\ V_L = L \frac{di}{dt} \\ V_R = iR \end{cases}$$

$$\Rightarrow L \frac{di}{dt} + Ri + \frac{q}{C} = 0 \quad \text{with} \quad i = \frac{dq}{dt} \quad \rightarrow \quad L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$$

$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{q}{LC} = 0}$$

The 2<sup>nd</sup> order differential equation for the RLC serial circuit

This equation has to be compared to the 2<sup>nd</sup> order equation for the forced oscillator

$$\boxed{\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0}$$

From term to term comparison, one can make the following analogy:

Mechanic	Electromagnetic
<b>x</b> –elongation	<b>q</b> –electric charge
<b>b</b> – damping constant	<b>R</b> –electric resistance
<b>m</b> – mas (inertia to movement)	<b>L</b> –inductance
<b>k</b> –elastic constant	<b>1/C</b> –inverse electric capacity

Mechanic	Electromagnetic
<b>x</b> –elongation	<b>q</b> –electric charge
<b>b</b> – damping constant (mechanic dissipation)	<b>R</b> –electric resistance (electric dissipation)
<b>m</b> – mass (inertia to changes of velocity)	<b>L</b> –inductance (opposes/inertia to changes of current)
<b>k</b> –elastic constant	<b>1/C</b> –inverse electric capacity
$v=dx/dt$	$i=dq/dt$
<b>K</b> $=mv^2/2$ stored in m	<b>E<sub>m</sub></b> $=Li^2/2$ magnetic field energy stored in L
<b>U</b> $=kx^2/2$ stored in spring (k)	<b>E<sub>el</sub></b> $=q^2/2C$ electric field energy stored in C

$$x(t) = Ae^{-\frac{b}{2m}t} \cos(\omega't + \phi)$$

$$q(t) = q_0 e^{-\frac{R}{2L}t} \cos(\omega't + \phi)$$

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \delta^2}$$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} = \sqrt{\omega_0^2 - \delta^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \delta = \frac{b}{2m}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \quad \text{and} \quad \delta = \frac{R}{2L}$$



## Quality factor

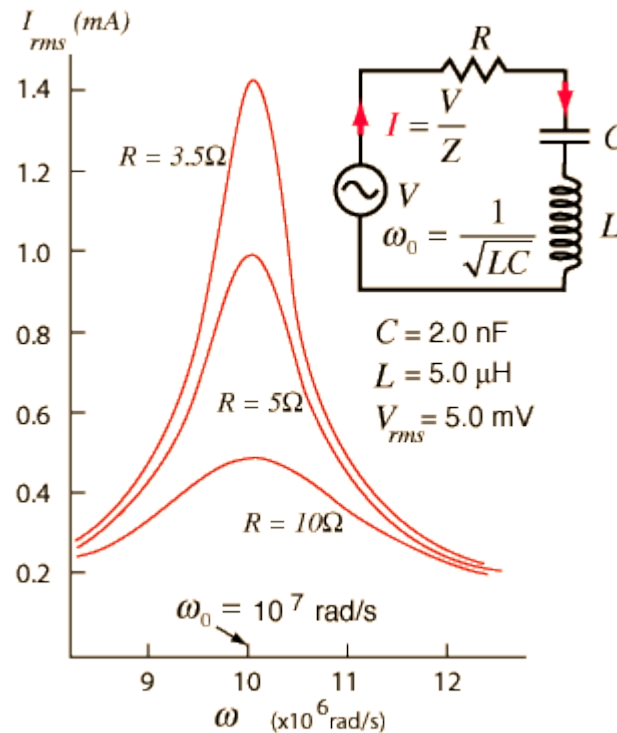
$$Q = \frac{2\pi}{T} \tau = \omega_0 \tau = \frac{\omega_0}{\delta} = \sqrt{\frac{km}{b}} = \sqrt{\frac{k}{m} \frac{m}{b}} = \omega_0 \frac{m}{b}$$

$$Q = \frac{2\pi}{T} \tau = \omega_0 \tau = \frac{\omega_0}{\delta} = \sqrt{\frac{L}{RC}} = \sqrt{\frac{1}{LC} \frac{L}{R}} = \omega_0 \frac{L}{R}$$

Large  $Q$  means small  $b$  and large  $km$

Large  $Q$  means small  $R$  and large  $L/C$

*All concepts related to damped, forced oscillation/resonance apply*



## Forced oscillations/resonance

### Mechanic

Driving external force

$$F(t) = F_{\max} \cos \omega_D t$$

$$\sum F = ma_x \Rightarrow -kx - b \frac{dx}{dt} - F_{\max} \cos \omega_d t = m \frac{d^2 x}{dt^2}$$

$$\frac{d^2 x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x + \frac{F_{\max}}{m} \cos \omega_d t = 0$$

$$A = \frac{F_{\max}}{\sqrt{(k - m\omega_d^2)^2 + b^2 \omega_d^2}}$$

$$\tan \phi = \frac{\frac{b}{m} \omega_d}{\omega_d^2 - \omega_0^2}$$

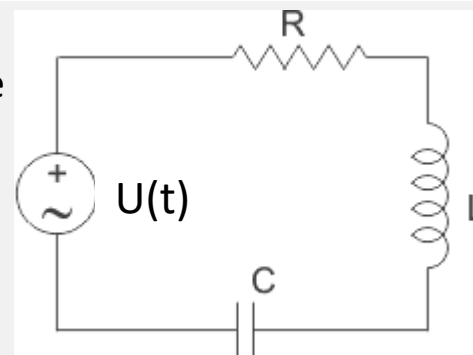
### Electromagnetic

Source of alternative voltage

$$U(t) = U_{\max} \cos \omega_D t$$

$$V_C + V_R + V_L + U = 0$$

$$\frac{q}{C} + iR + L \frac{di}{dt} + U = 0$$



$$\frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q + \frac{U_{\max}}{L} \cos \omega_d t = 0$$

$$q_{\max} = \frac{U_{\max}}{\sqrt{L \left( \frac{1}{LC} - \omega_d^2 \right)^2 + R^2 \omega_d^2}} = \frac{U_{\max}}{\sqrt{L (\omega_0^2 - \omega_d^2)^2 + R^2 \omega_d^2}}$$

$$\tan \phi = \frac{\frac{R}{L} \omega_d}{\omega_d^2 - \frac{1}{LC}}$$

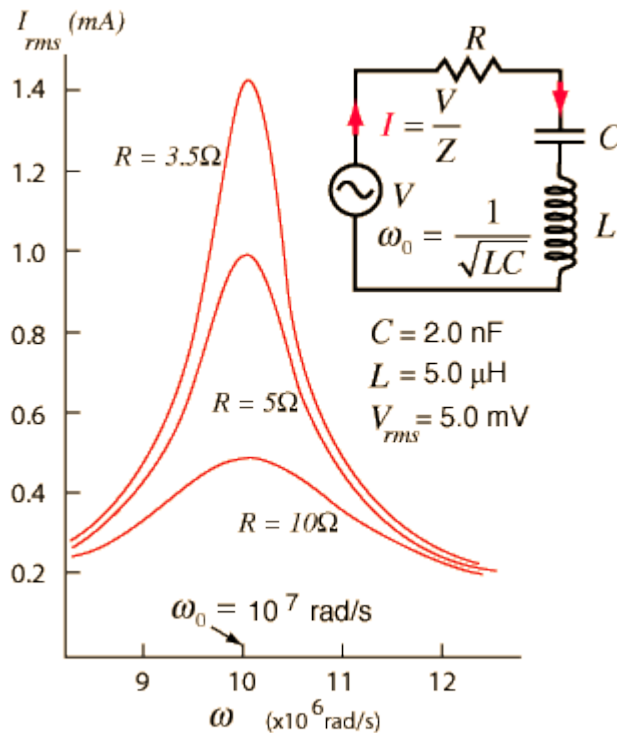
In such RLC circuit, the current intensity  $i(t)$  will be:

$$i = \frac{dq}{dt} = \omega_d q_{\max} \sin(\omega_d t + \phi) = i_0 \sin(\omega_d t + \phi)$$

$$\text{where: } i_0 = \omega_d q_{\max} = \frac{U_{\max}}{\sqrt{R^2 + \left(\frac{1}{\omega_d C} - \omega_d L\right)^2}}$$

The current will have a maximum for:

$$\frac{1}{\omega_d C} = \omega_d L \Rightarrow \omega_d = \frac{1}{\sqrt{LC}} = \omega_0$$



So at resonance when the external voltage frequency became equal to the frequency of the undamped proper oscillations

At resonance, the amplitude of current oscillations is fully related to the resistance value

# Simple harmonic oscillation

Mechanic

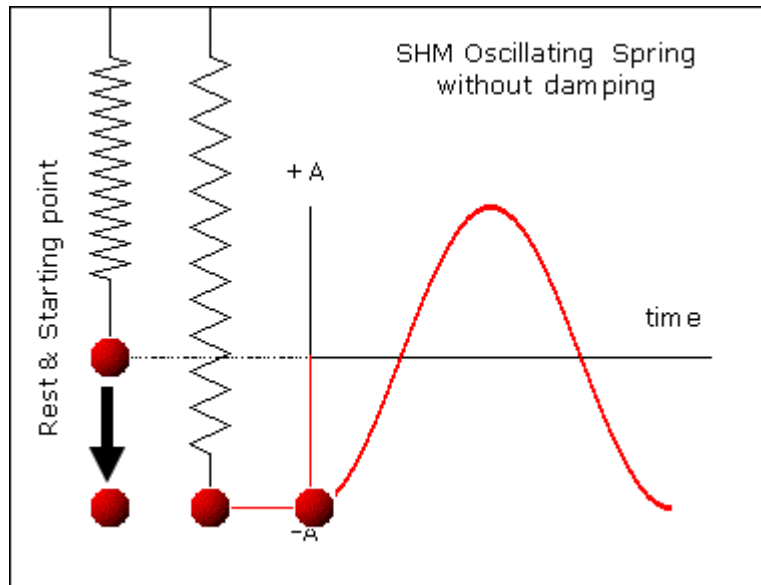
Electromagnetic

Mechanic SHM oscillator

Electrical oscillator LC

$b=0$  no mechanical dissipation

$R=0$  no electrical dissipation



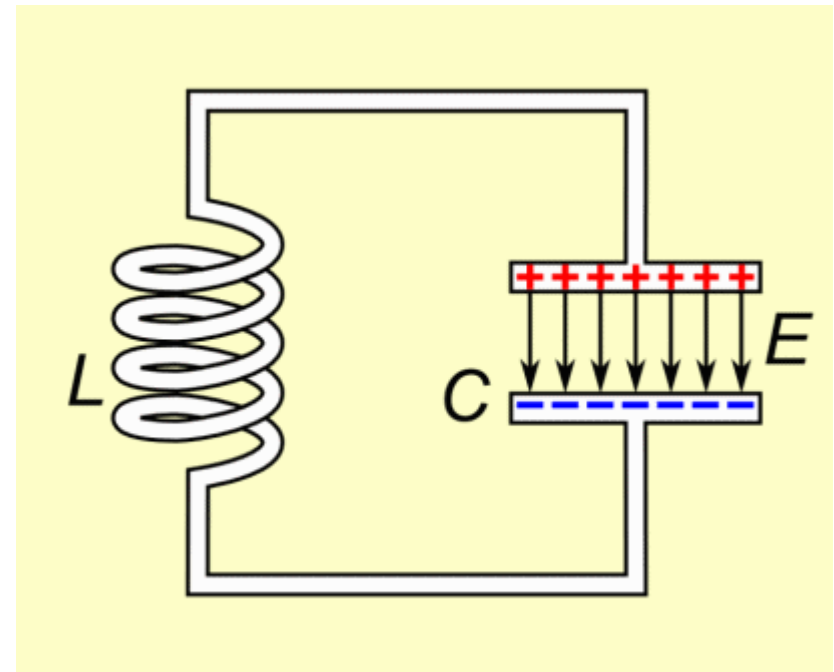
$$k \leftrightarrow 1/C$$

$$m \leftrightarrow L$$

$$K \leftrightarrow U$$

Continuous conversion

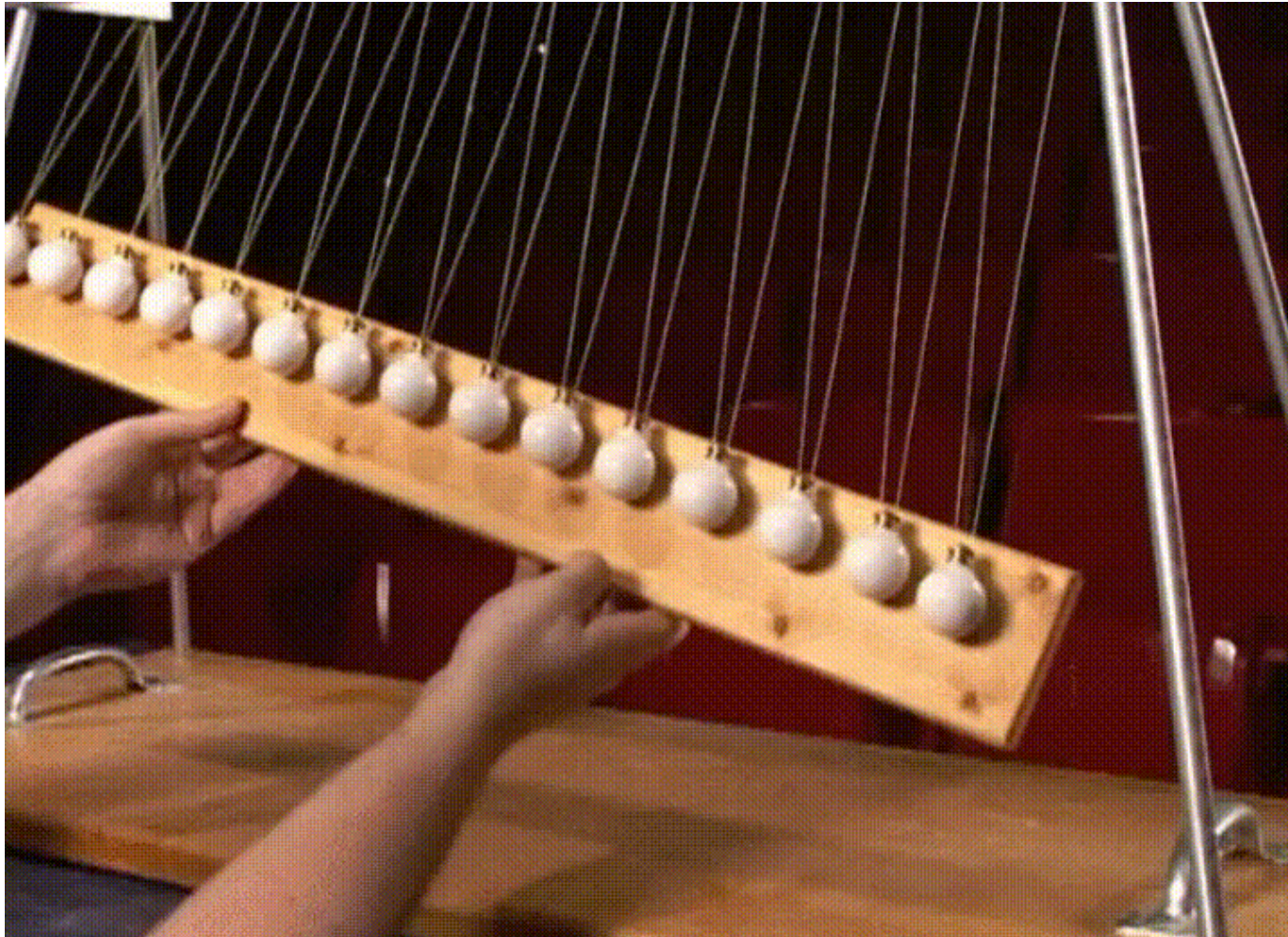
$$mv^2/2 \leftrightarrow kx^2/2$$



$$E_m \leftrightarrow E_{el}$$

$$LI^2/2 \leftrightarrow q^2/(2C)$$

## Pendulum wave, **Kinetic art**



Use the **simple pendulum motion** to create a “**pendulum wave apparatus**”: a device where many pendulums of different lengths (and therefore different **periods**) start swinging at the same time. As they move in and out of sync, the pendulums create a sequence of cycling visual wave patterns.